The adiabatic theorem and Berry's phase

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A study is presented of Berry's observation that when a quantum-mechanical system is transported on a closed adiabatic journey, a topological phase arises in addition to the usual dynamical phase expected from the adiabatic theorem. Consequences are explored in the case of a simple magnetic moment–magnetic field interaction and are shown to lead, among other things, to Dirac's famous relation between electric and magnetic charges.

I. INTRODUCTION

It is now over 60 years since the development of quantum mechanics, and the subject is by now an integral part of every graduate and undergraduate physics curriculum. It is remarkable then that in 1984 Berry pointed out a feature that had been overlooked by others for these 60 years, having to do with the existence of a topological phase factor that can arise in certain applications of the adiabatic theorem.[1] In the 5 years since this discovery, many authors have studied this phenomenon and applications to a remarkably wide range of problems have been discussed—in particle physics and quantum field theory, in condensed matter physics, in atomic and molecular physics, etc.[2] While many such articles appear somewhat formal and forbidding, the concept itself is elementary enough to be presented to an advanced quantum mechanics class. (Indeed, Berry's original article is remarkable for its clarity and should be read by anyone interested in this subject, as should a recent popular account which he has written.[1]) Below then we outline Berry's idea, with application made to a simple physical system.

The adiabatic approximation,[3] wherein the time scale over which a (time-dependent) Hamiltonian varies is long compared to typical quantum-mechanical oscillation periods, asserts that if a system begins at time t_i in an instantaneous eigenstate $\psi_n(\mathbf{x},t_i)$, then at all later times it will remain in this same eigenstate, but develops a simple dynamical phase factor

$$\exp - i \int_{t_i}^{t_f} dt\, E_n(t) . \tag{1}$$

Here $E_n(t)$ and $\psi_n(\mathbf{x},t)$ are "instantaneous" eigenvalues, eigenfunctions of the time-dependent Hamiltonian $h(t)$, satisfying

$$h(t)\psi_n(\mathbf{x},t) = E_n(t)\psi_n(\mathbf{x},t) . \tag{2}$$

It is usually noted in deriving this result that one is free to adjust the phase of the instantaneous eigenstates arbitrarily. However, this is *not* always the case. Specifically in circumstances wherein at time $t = t_f$ the Hamiltonian $h(R_i(t))$ of a system returns, after an adiabatic excursion, to its form at $t = t_i$, there can arise an additional *topological* phase factor

$$\exp i\Phi_n , \tag{3}$$

where

$$\Phi_n = i \oint dR_i \cdot \langle \psi_n(\mathbf{R}_i) | \nabla_{R_i} \psi_n(\mathbf{R}_i) \rangle . \tag{4}$$

Here, $R_i(t)$ denotes a time-dependent set of parameters—$i = 1,2,...,k$—on which the Hamiltonian depends. When $k = 1$, we have a simple time-dependent Hamiltonian and

$$\Phi_n = 0 , \tag{5}$$

in agreement with the naive adiabatic theorem. It is the existence of such a topological factor which was pointed out by Berry,[1] and consequently Φ_n is called Berry's phase.

In the next section we formally derive the Berry phase and discuss its significance. In Sec. III we examine a simple application to a magnetic moment–magnetic field interaction. Finally, in Sec. IV we summarize our results.

II. BERRY'S PHASE DERIVATION

Consider a Hamiltonian

$$h(R_i(t)), \quad i = 1,2,...,k ,$$

which depends on a set of k time-dependent parameters $R_i(t)$. An example could be dependence on a time-dependent vector field, in which case $k = 3$ and the parameters are the three independent components of the field vector. However, we shall deal here with the general case.

We suppose that the rate of change of the parameters $R_i(t)$ is much slower than a typical orbital frequency $\Delta E_n(t)$ so that the adiabatic condition obtains. Then the adiabatic theorem asserts that a system which begins at time t_i in instantaneous eigenstate $\psi_n(\mathbf{x},t_i)$ will evolve almost completely into eigenstate $\psi_n(\mathbf{x},t_f)$ at time t_f. Here we wish to look at the phase factor which accompanies $\psi_n(\mathbf{x},t_f)$. Writing the general solution to the Schrödinger equation as $\psi(\mathbf{x},t)$ with

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = h(R_i(t))\psi(\mathbf{x},t) , \tag{6}$$

we define

$$\psi(\mathbf{x},t) \equiv \psi_n(\mathbf{x};t)\exp\left(- i \int_{t_i}^{t} dt'\, E_n(t')\right)\exp i\gamma_n(t) , \tag{7}$$

and find

$$\dot\gamma_n(t) = i \int d^3x\, \psi_n^*(\mathbf{x},t)\dot\psi_n(\mathbf{x},t) . \tag{8}$$

Since ψ_n achieves its time dependence only from the existence of the parameters $R_i(t)$ [i.e., $\psi_n(\mathbf{x},t)$ would be time *independent* were $R_i = \text{const}$], we may write

$$\psi_n(\mathbf{x};t) \equiv \psi_n(\mathbf{x},R_i(t)) , \tag{9}$$

and

$$\dot\gamma_n(t) = i \int d^3x \sum_i \psi_n^*(\mathbf{x},R_i(t))\nabla_{R_i}\psi_n(\mathbf{x},R_i(t))\, \dot R_i(t) . \tag{10}$$

With

$$\mathbf{R}(t) = \begin{pmatrix} R_1(t) \\ R_2(t) \\ \vdots \\ R_k(t) \end{pmatrix} \quad (11)$$

as a k-component column vector, we can express the result in the simplified notation

$$\dot{\gamma}_n(t) = i\langle n;\mathbf{R}(t)|\nabla_R n;\mathbf{R}(t)\rangle\cdot\dot{\mathbf{R}}(t) . \quad (12)$$

Thus far, there is nothing new, and the existence of this phase $\gamma_n(t)$ in addition to the usual dynamical phase factor

$$\exp -i\int_{t_i}^{t} dt'\, E_n(R_i(t)) \quad (13)$$

has been known for a long time.[1] It was generally assumed that $\gamma_n(t)$ could be eliminated by redefining the (undetermined) phase of the eigenstate $|n;\mathbf{R}(t)\rangle$. Berry, however, realized that such a phase *is* observable when the time evolution brings the parameter vector $\mathbf{R}(t)$ back to its starting point—i.e., $\mathbf{R}(t_f) = \mathbf{R}(t_i)$—whereby the state vector $|n;\mathbf{R}(t_f)\rangle$ can be interfered with $|n;\mathbf{R}(t_i)\rangle$. *This* quantity, which may be written as

$$\gamma_n = i\int_{t_i}^{t_f} dt\, \dot{\mathbf{R}}(t)\cdot\langle n;\mathbf{R}(t)|\nabla_R n;\mathbf{R}(t)\rangle$$

$$= i\oint d\mathbf{R}\cdot\langle n;\mathbf{R}(t)|\nabla_R n;\mathbf{R}(t)\rangle \quad (14)$$

is called Berry's phase and is an observable. That γ_n is a physical parameter can be further emphasized by expressing this result in the suggestive notation

$$\mathbf{A}_n(\mathbf{R}) = i\langle n;\mathbf{R}|\nabla_R n;\mathbf{R}\rangle , \quad (15)$$

so that

$$\gamma_n = \oint d\mathbf{R}\cdot\mathbf{A}_n(\mathbf{R}) \quad (16)$$

is written in terms of a "vector potential" like quantity. If we choose to redefine the phase of the eigenstate by an arbitrary phase $\phi(\mathbf{R})$,

$$|n;\mathbf{R}\rangle \to \exp i\phi(\mathbf{R})|n;\mathbf{R}\rangle , \quad (17)$$

then

$$\mathbf{A}_n(\mathbf{R}) \to \mathbf{A}_n(\mathbf{R}) - \nabla_R\phi(\mathbf{R}) , \quad (18)$$

which is analogous to a gauge transformation. Obviously, an observable cannot depend upon the choice of gauge, and it is clear that Berry's phase obeys this property since, by Stokes' theorem (suitably generalized if $k \neq 3$), we can write

$$\gamma_n = \int d\mathbf{R}\cdot\mathbf{A}_n(\mathbf{R}) = \int d\mathbf{S}\cdot\nabla_R\times\mathbf{A}_n(\mathbf{R})$$

$$\to \int d\mathbf{S}\cdot\nabla_R\times[\mathbf{A}_n(\mathbf{R}) - \nabla_R(\mathbf{R})]$$

$$= \int d\mathbf{S}\cdot\nabla_R\times\mathbf{A}_n(\mathbf{R}) , \quad (19)$$

so that γ_n is unchanged.

A physical significance may be ascribed to this "vector potential" if the parameters $R_i(t)$ are themselves quantized. That is, suppose the full Hamiltonian for some system is

$$H = \mathbf{P}^2/2M + \mathbf{p}^2/2m + V(\mathbf{R},\mathbf{r}) , \quad (20)$$

where

$$h(\mathbf{R}) = \mathbf{p}^2/2m + V(\mathbf{R},\mathbf{r}) \quad (21)$$

is the simple time-dependent Hamiltonian of Eq. 6. Then, writing the full wavefunction as

$$|n;\mathbf{R}(t)\rangle\chi(\mathbf{R}(t)) = |\psi(\mathbf{R},\mathbf{r})\rangle , \quad (22)$$

we find

$$H|\psi(\mathbf{R},\mathbf{r})\rangle = E_n(\mathbf{R})|\psi(\mathbf{R},\mathbf{r})\rangle + |n;\mathbf{R}(t)\rangle(\mathbf{P}^2/2M)\chi(\mathbf{R}) - i\nabla_R|n;\mathbf{R}(t)\rangle\cdot(1/M)\mathbf{P}\chi(\mathbf{R}) - (1/2M)\nabla_R^2|n;\mathbf{R}(t)\rangle\chi(\mathbf{R}) . \quad (23)$$

Finally, projecting out the $|n;\mathbf{R}(t)\rangle$ states and neglecting off-diagonal matrix elements ($\langle n';\mathbf{R}(t)|\nabla_R|n;\mathbf{R}(t)\rangle \approx 0$), we find that $\chi(\mathbf{R})$ obeys a Schrödinger equation with the effective Hamiltonian

$$H_{\text{eff}} = (1/2M)(\mathbf{P} - \mathbf{A})^2 + U(\mathbf{R}) , \quad (24)$$

where

$$U(\mathbf{R}) = E_n(\mathbf{R}) - (1/2M)[\langle n;\mathbf{R}(t)|\nabla_R^2\, \mathbf{n};\mathbf{R}(t)\rangle + \mathbf{A}_n^2(\mathbf{R}(t))] . \quad (25)$$

This, of course, is simply the Born–Oppenheimer approach to a system containing both fast ($\mathbf{r}$, $\mathbf{p}$) and slow ($\mathbf{R}$, $\mathbf{P}$) degrees of freedom.[4] We observe that the fast system affects the dynamics of its slow counterpart via the potential energy $U(\mathbf{R})$ and a "vector potential" $\mathbf{A}_n(\mathbf{R})$. According to Eq. (16), Berry's phase is simply the line integral of this vector potential which, by Stokes' theorem, can be rewritten in terms of an integral

$$\gamma_n = \int \mathbf{B}_n(\mathbf{R})\cdot d\mathbf{S} \quad (26)$$

over a surface bounded by the curve $\oint d\mathbf{R}$. Here $\mathbf{B}_n(\mathbf{R})$ is a fieldlike quantity

$$\nabla_R\times\mathbf{A}_n(\mathbf{R}) = \mathbf{B}_n(\mathbf{R}) , \quad (27)$$

and Berry's phase becomes the flux of this field through the surface. The field $\mathbf{B}(\mathbf{R})$ will have a nontrivial structure in the presence of sources, which occur when two or more of the "fast" eigenvalues $|n;\mathbf{R}(t)\rangle$ become degenerate for some value of $\mathbf{R}(t)$. This is clear since

$$\gamma_n = i\int ds\cdot\nabla_R\times\langle n;\mathbf{R}(t)|\nabla_R n;\mathbf{R}(t)\rangle$$

$$= -\text{Im}\int d\mathbf{S}\cdot\langle\nabla_R n;\mathbf{R}(t)|\times|\nabla_R n;\mathbf{R}(t)\rangle . \quad (28)$$

Now insert a complete set of intermediate states

$$\mathbf{1} = \sum |m;\mathbf{R}(t)\rangle\langle m;\mathbf{R}(t)| , \quad (29)$$

and note that the diagonal contribution vanishes since $\langle n;\mathbf{R}(t)|\nabla_R n;\mathbf{R}(t)\rangle$ is purely imaginary.[5] Then, noting[6]

$$\langle m;\mathbf{R}(t)|\nabla_R n;\mathbf{R}(t)\rangle = \frac{\langle m;\mathbf{R}(t)|\nabla_R h\,|n;\mathbf{R}(t)\rangle}{E_n - E_m}\, m\neq n\,, \tag{30}$$

we have

$$\mathbf{B}_n(\mathbf{R}) = -\,\mathrm{Im}\sum_{m\neq n}\frac{\langle n;\mathbf{R}(t)|\nabla_R h\,|m;\mathbf{R}(t)\rangle\times\langle m;\mathbf{R}(t)|\nabla_R h\,|n;\mathbf{R}(t)\rangle}{(E_m - E_n)^2}\,. \tag{31}$$

The existence of a degeneracy implies an infinity in $\mathbf{B}(\mathbf{R})$ and thus the presence of a field source. The Berry phase is the flux associated with such sources.

III. BERRY'S PHASE—AN EXAMPLE

While one could continue to develop this formalism, it is useful to give an example wherein the formalism can be applied. Perhaps the simplest is that of a spin-$\frac{1}{2}$ particle in an external magnetic field "$\mathbf{R}(t)$" for which the relevant Hamiltonian is[1]

$$h(\mathbf{R}(t)) = -(\mu/2)\,\boldsymbol{\sigma}\cdot\mathbf{R}(t) = -\frac{\mu}{2}\begin{pmatrix} Z(t) & X(t)-iY(t)\\ X(t)+iY(t) & -Z(t)\end{pmatrix}. \tag{32}$$

[Note we have written the external magnetic field as $\mathbf{R}(t)$ so as not to be confused with the curl of the Berry potential which we have denoted by the symbol $\mathbf{B}(t)$.] The eigenvalues are given, of course, by

$$E_+(\mathbf{R}) = -E_-(\mathbf{R}) = +(\mu/2)[X^2(t)+Y^2(t)+Z^2(t)]^{1/2}\,, \tag{33}$$

so that there exists a degeneracy when $\mathbf{R}=0$. Then since

$$\nabla_R h(\mathbf{R}(t)) = -(\mu/2)\sigma\,, \tag{34}$$

we observe that, picking $\mathbf{R}$ along the z axis,

$$\mathbf{B}_\uparrow(R\hat{\mathbf{k}}) = -\,\mathrm{Im}\langle\uparrow|\sigma|\downarrow\rangle\times\langle\downarrow|\sigma|\uparrow\rangle\frac{1}{4R^2(t)} = -\left[\frac{\hat{\mathbf{k}}}{4R^2(t)}\right]\mathrm{Im}(\langle\uparrow|\sigma_x|\downarrow\rangle\langle\downarrow|\sigma_y|\uparrow\rangle - \langle\uparrow|\sigma_y|\downarrow\rangle\langle\downarrow|\sigma_x|\uparrow\rangle) = -\hat{\mathbf{k}}/2R^2(t)\,, \tag{35}$$

or, in general,

$$\mathbf{B}_\uparrow(\mathbf{R}) = -\hat{\mathbf{R}}/2R^2(t)\,, \tag{36}$$

which corresponds to a "magnetic monopole" of strength $\frac{1}{2}$ located at the origin (i.e., at the place of the degeneracy). The Berry phase is the flux associated with this monopole through the surface which is circumnavigated in parameter space and is given by

$$\gamma_\uparrow = \pm\tfrac{1}{2}\Delta\Omega\,, \tag{37}$$

where $\Delta\Omega$ is the solid angle subtended by the closed path as seen from the origin of parameter space and the $\pm$ refers to the direction in which the line integration is traversed.

This may be verified explicitly by using the representation[7]

$$|\uparrow,\mathbf{R}\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)e^{i\phi}\end{pmatrix}, \tag{38}$$

for a spinor along the direction $\hat{\mathbf{R}}$ specified by spherical coordinate angles θ and ϕ. Then from Eq. (15) the associated $\mathbf{A}(\mathbf{R})$ is given by

$$\mathbf{A}_\uparrow(\mathbf{R}) = i\langle\uparrow;\mathbf{R}|\nabla_R|\uparrow;\mathbf{R}\rangle = -\hat{\mathbf{a}}_\phi\frac{1}{R\sin\theta}\sin^2\frac{\theta}{2} = -\hat{\mathbf{a}}_\phi\frac{1}{2R}\tan\frac{\theta}{2}\,, \tag{39}$$

which is the "vector potential" of a magnetic monopole of strength $\frac{1}{2}$ located at $\mathbf{R}=0$.

Note:

$$\nabla\times\mathbf{A}(\mathbf{R}) = -\hat{\mathbf{a}}_r\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = -\hat{\mathbf{a}}_r\frac{1}{2R^2}\,. \tag{40}$$

Taking a path in parameter space at fixed θ from θ, ϕ to $\theta, \phi+2\pi$, we find the Berry phase to be

$$\gamma_\uparrow = \oint\mathbf{A}_\uparrow\cdot d\mathbf{R} = \pm A_\phi 2\pi R\sin\theta = \pm\pi(1-\cos\theta)\,. \tag{41}$$

The solid angle swept out by this trajectory is clearly

$$\Delta\Omega = \int_0^\theta \sin\theta\, d\theta\int_0^{2\pi}d\phi = 2\pi(1-\cos\theta)\,, \tag{42}$$

so that

$$\gamma_\uparrow = \pm\tfrac{1}{2}\Delta\Omega\,, \tag{43}$$

as expected from Eq. (37).

This result has a number of interesting consequences. One is that the effect can be measured. Thus consider a beam of neutrons which is split into two components. One traverses a constant magnetic field $R_0\hat{\mathbf{k}}$. The second follows a magnetic field of the same magnitude, but which slowly rotates about a cone with semiangle θ, as described above, and returns to its original position. If then the beams are recombined, the intensity should vary as

$$I(\theta) = \tfrac{1}{2}I_0|1+e^{i\gamma_\uparrow}|^2 = I_0\cos^2\tfrac{1}{2}\gamma_\uparrow = I_0\cos^2\tfrac{1}{2}\pi(1-\cos\theta)\,. \tag{44}$$

This experiment has been performed and the results are exactly as predicted.[8]

A second interesting consequence of Berry's phase is that it enables a simple understanding of Dirac's argument that magnetic charge must be quantized in inverse units of electric charge.[9] Thus consider the surface integral shown in Fig. 1. One has the choice of employing either the surface S_1 *or* S_2. But these must yield the *same* physics. Hence

$$\int_{S_1}\mathbf{B}\cdot d\mathbf{S} = \int_{S_2}\mathbf{B}\cdot d\mathbf{S} + 2\pi p, \quad p = 0, \pm 1, \ldots\,. \tag{45}$$

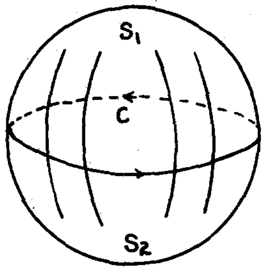

Fig. 1. When integrating over a surface bounded by a curve C as shown, one has the choice of using either surface S_1 or S_2 in order to calculate the relevant flux.

Noting the orientation of the normals, we subtract to yield

$$\int \mathbf{B}\cdot d\,\mathbf{S} = 2\pi p\,. \tag{46}$$

[In the case studied above—Eq. (36)— we have $p = +1$. However, it is straightforward to show that if one starts with a system with spin S and projection S_z that $p = 2S_z$.] In the simple electromagnetic case of the interaction between an electric charge and magnetic monopole g this condition reads[10]

$$4\pi eg = 2\pi p\,,$$

i.e.,

$$eg = \tfrac{1}{2}p\,, \tag{47}$$

which is Dirac's constraint. Note that Eq. (47) implies that magnetic charge is quantized in units of $1/2e$. Or, turning the argument around, if there exists a monopole of strength g anywhere in the universe, then electric changes must have values

$$e = p/2g, \quad p = 0, \pm 1, \ldots\,. \tag{48}$$

This is the only argument of which I am aware that "explains" the experimental observation of quantization of electric charge.

There is one other aspect here which deserves comment and that is the relation between the Berry potential and rotational invariance. Specifying an arbitrary rotation by angle α and axis $\hat{\mathbf{n}}$, it is well known that the operator

$$O = \exp(i\mathbf{L}\cdot\hat{\mathbf{n}}\delta\alpha), \quad \text{with } \mathbf{L} = \mathbf{R}\times\mathbf{P}\,, \tag{49}$$

is the generator of rotations.[11] Taking a rotationally invariant Hamiltonian such as $H_{\text{eff}} = \mathbf{P}^2/2M + U(r)$, we have

$$O\,H_{\text{eff}}\,O^{-1} = (\mathbf{P} + \delta\alpha\,\hat{\mathbf{n}}\times\mathbf{p})^2/2M + U(r) = H_{\text{eff}}\,, \tag{50}$$

which implies

$$[\mathbf{L}, H_{\text{eff}}] = 0\,. \tag{51}$$

That is, rotational invariance of the Hamiltonian is equivalent to conservation of angular momentum, as required by Noether's theorem.[12]

Now if the Berry potential $\mathbf{A}(\mathbf{R})$ were an ordinary vector field such that under the infinitesimal rotation

$$\mathbf{R} \to \mathbf{R} + \delta\alpha\,\hat{\mathbf{n}}\times\mathbf{R}\,, \tag{52}$$

we have

$$\mathbf{A}(\mathbf{R}) \to \mathbf{A}(\mathbf{R} - \delta\alpha\,\hat{\mathbf{n}}\times\mathbf{R}) + \delta\alpha\,\hat{\mathbf{n}}\times\mathbf{A}(\mathbf{R})\,, \tag{53}$$

then rotational invariance of H_{eff} and hence the conservation of total angular momentum would be transparent. Such a transformation property is guaranteed, for example, by the choice

$$\mathbf{A}(\mathbf{R}) = \hat{\mathbf{R}}f(R)\,. \tag{54}$$

Obviously, the monopole Berry potential—Eq. (39)—is not of this form and thus simple rotational invariance does not obtain. Nevertheless, the *physics* can still be invariant, provided that the new vector potential is related to the old by a gauge transformation[13]

$$\mathbf{A}(\mathbf{R}) \to \mathbf{A}(\mathbf{R}) + \nabla_R\phi(\mathbf{R})\,. \tag{55}$$

This condition then reads

$$\begin{aligned}\mathbf{A}(\mathbf{R}) &\to \mathbf{A}(\mathbf{R}) + \delta\alpha\,\hat{\mathbf{n}}\times\mathbf{A}(\mathbf{R}) - \delta\alpha\,\hat{\mathbf{n}}\times\mathbf{R}\cdot\nabla_R\mathbf{A}(\mathbf{R}) \\ &= \mathbf{A}(\mathbf{R}) + \nabla_R\phi(\mathbf{R})\,,\end{aligned} \tag{56}$$

i.e.,

$$\nabla_R\phi(\mathbf{R}) = \delta\alpha[\hat{\mathbf{n}}\times\mathbf{A}(\mathbf{R}) - \hat{\mathbf{n}}\times\mathbf{R}\cdot\nabla_R\mathbf{A}(\mathbf{R})]\,. \tag{57}$$

This solution to this equation for our case is

$$\phi(\mathbf{R}) = -\delta\alpha((1/2R)\mathbf{R}\cdot\hat{\mathbf{n}} + \hat{\mathbf{n}}\cdot\mathbf{R}\times\mathbf{A})\,. \tag{58}$$

[This is clear since

$$\begin{aligned}\nabla_R\phi(\mathbf{R}) &= \delta\alpha\left(-\frac{1}{2}\frac{\hat{\mathbf{n}}}{R} + \frac{1}{2R}\hat{\mathbf{R}}\hat{\mathbf{n}}\cdot\hat{\mathbf{R}} + \hat{\mathbf{n}}\times\mathbf{A} - (\hat{\mathbf{n}}\times\mathbf{R})_i\nabla_R A_i\right) \\ &= \delta\alpha(-(1/2R)\hat{\mathbf{R}}\times(\hat{\mathbf{n}}\times\hat{\mathbf{R}}) + \hat{\mathbf{n}}\times\mathbf{A} \\ &\quad - (\hat{\mathbf{n}}\times\mathbf{R})\times(\nabla_R\times\mathbf{A}) - \hat{\mathbf{n}}\times\mathbf{R}\cdot\nabla_R\mathbf{A})\,.\end{aligned} \tag{59}$$

Using $\nabla_R\times\mathbf{A} = \hat{\mathbf{R}}/2\mathrm{R}^2$ the result follows.]

Writing the rotation operator now as

$$O' = \exp i\,\delta\alpha\,\hat{\mathbf{n}}\cdot\mathbf{r}\times(\mathbf{P} - \mathbf{A})\,, \tag{60}$$

we find

$$\begin{aligned}&O'\frac{(\mathbf{p}-\mathbf{A})^2}{2M}O' \\ &= \frac{[\mathbf{P} - \mathbf{A} - i\,\delta\alpha\,\hat{\mathbf{n}}\times(\mathbf{P} - \mathbf{A}) - i\,\delta\alpha(\hat{\mathbf{n}}\times\mathbf{r})\times(\nabla\times\mathbf{A})]^2}{2M} \\ &= \frac{(\mathbf{P}-\mathbf{A})^2}{2M} - 2i\,\delta\alpha\,\frac{(\mathbf{P}-\mathbf{A})\cdot(\hat{\mathbf{n}}\times\mathbf{r})\times(\nabla\times\mathbf{A})}{2M}\,,\end{aligned} \tag{61}$$

so that the Hamiltonian $H = (\mathbf{P} - \mathbf{A})^2/2M$ is *not* invariant. In order to cancel the offending term, we must modify the rotation operator via

$$O' \to O'' = \exp i\,\delta\alpha\,\hat{\mathbf{n}}\cdot[\mathbf{R}\times(\mathbf{P} - \mathbf{A}) - \tfrac{1}{2}\hat{\mathbf{R}}]\,. \tag{62}$$

Then

$$O''[(\mathbf{P}-\mathbf{A})^2/2M]O''^{-1} = (\mathbf{P}-\mathbf{A})^2/2M\,, \tag{63}$$

as desired. The physics behind this change is easily identified by noting that

$$\mathbf{J} = \mathbf{R}\times M\dot{\mathbf{R}} - \tfrac{1}{2}\hat{\mathbf{R}} \qquad (64)$$

is then a conserved angular momentum. The first term here is simply the usual rotational (kinetic) angular momentum. The second is not as familiar, although it too has a simple origin—it is the angular momentum contained in the **E**,**B** fields of a charge e monopole g system with $eg = \frac{1}{2}$. Thus place the monopole at the origin of coordinates and the charge at height a along the z axis. Then

$$\mathbf{E} = e(\mathbf{r} - a\hat{\mathbf{k}})/|\mathbf{r} - a\hat{\mathbf{k}}|^3,$$
$$\mathbf{B} = g\mathbf{r}/r^3, \qquad (65)$$

and the field angular momentum is found to be

$$\mathbf{J}_{\text{field}} = \frac{1}{4\pi}\int d^3r\,\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$$
$$= \frac{gea}{4\pi}\int d^3r\frac{1}{r^3}\frac{1}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}}$$
$$\times(r^2\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{k}}r^2)$$
$$\times(r^2\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{k}}r^2)$$
$$\times\frac{1}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}}(\cos^2\theta - 1)$$
$$\times\frac{1}{(r^2 + a^2 - 2ra\cos\theta)^{3/2}}(\cos^2\theta - 1)$$
$$= -\hat{\mathbf{k}}ge. \qquad (66)$$

Thus **J** is simply the total angular momentum of the system (rotation plus field) as stated.

Note that in our case the field angular momentum is found to be $\frac{1}{2}\hbar$. This means that a system of a simple electric charge and monopole is a fermion! This result is well known, but is easily seen from perspective. Such a phenomenon is also found in quantum field theory wherein a system of spin-zero fields can be found to have half-integral angular momentum.[14] Such "skyrmions" are currently popular pictures of the structure of the nucleon and are associated with the existence of so-called "anomalies," wherein classical symmetries are violated upon quantization.[15] The quantum-mechanical example above provides an interesting and elementary introduction to these fascinating topics.

While we have above explored the consequences of Berry's phase in a single simple system, numerous additional applications have by now been pointed out. In fact, the first realization of the need for such "anomalous" phases was noted long ago by chemists studying molecular structure, who had emphasized the need to modify the slow Hamiltonian with vector potential type terms.[16] Subsequent to Berry's work, the concept of adiabatic phases has been used in order to understand effects in various areas of physics, such as the quantized Hall effect,[17] the spin-statistics properties of quasiparticle excitations in two-dimensional systems,[18] the rotation of photon polarization in helical optical fibers,[19] etc. Even a classical analog, the Hannay angle, has been found, which gives a new degree of understanding to adiabatic classical rotations such as occur in the precession of the Foucault pendulum.[20] In nearly every case, what is found is not so much something new but rather an elegant and trenchant perspective on the underlying physics. We shall not here attempt to discuss any of these additional systems. However, a number of useful articles of a summary nature are available.[21]

IV. CONCLUSION

We have in this article examined the interesting result pointed out by Berry that when a quantum-mechanical system is moved adiabatically through a closed path in parameter space an additional topological phase can arise in addition to the expected dynamical phase factor

$$\exp -i\int_{t_i}^{t_f} dt\, E_n(t).$$

The meaning of this additional phase was found to be the "flux" emerging from the crossing point of the adiabatic energy levels.

We examined this phenomenon in the specific case of a simple interaction between a magnetic moment and external and adiabatically changing magnetic field. The Berry vector potential was found to be that for a magnetic monopole–charge interaction, and using this result, we were able to rederive Dirac's famous relation between electric and magnetic charge as well as some interesting physics associated with the angular momentum of such a system. While both results can be found by other means, the Berry potential argument is an elegant one and provides connections between otherwise apparently disparate areas of physics.

ACKNOWLEDGMENT

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[1]M. V. Berry, "Quantal Phase Factors Accompanying Adiabatic Changes," Proc. R. Soc. London Ser. A **392**, 45–57 (1984); "The Geometric Phase," Sci. Am. **259**(6), 46–52 (1988).

[2]For a nice summary of different applications see, e.g., R. Jackiw, "Berry's Phase—Topological Ideas from Atomic, Molecular and Optical Physics," Comments At. Mol. Phys. **XXI**, 71–82 (1988) and references therein.

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[5]This is clear since from the normalization condition we find

$$0 = \nabla_R\langle n;\mathbf{R}|n;\mathbf{R}\rangle = \langle n;\mathbf{R}|\nabla_R n;\mathbf{R}\rangle$$
$$\nabla_R\langle n;\mathbf{R}|n;\mathbf{R}\rangle = \langle n;\mathbf{R}|\nabla_R n;\mathbf{R}\rangle$$
$$= 2Re\langle n;\mathbf{R}|\nabla_R n;\mathbf{R}\rangle.$$

[6]This can be seen from eigenvalue condition

$$h(\mathbf{R})|n;\mathbf{R}\rangle = E_n(\mathbf{R})|n;\mathbf{R}\rangle.$$

Then

$$\nabla_R h(\mathbf{R})|n;\mathbf{R}\rangle + h(\mathbf{R})\nabla_R|n;\mathbf{R}\rangle$$
$$= E_n(\mathbf{R})\nabla_R|n;\mathbf{R}\rangle + \nabla_R E_n(\mathbf{R})|n;\mathbf{R}\rangle.$$

Projecting onto $\langle m;\mathbf{R}|$, we find

$$\langle m;\mathbf{R}|\nabla_R h(\mathbf{R})|n;\mathbf{R}\rangle$$
$$= [E_n(\mathbf{R}) - E_m(\mathbf{R})]\langle m;\mathbf{R}|\nabla_R n;\mathbf{R}\rangle$$

and the result follows.

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$$H = (\mathbf{P} - e\mathbf{A})^2/2M,$$

where A is the vector potential associated with monopole. Comparing with the Berry Hamiltonian—Eq. (24)—it is clear that the *parameter* space field $\mathbf{B}(\mathbf{R})$ corresponds to the "*real world*" case $eg\mathbf{B}(\mathbf{r})$.

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Nonradiating sources: The subtle art of changing light into black

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When a point charge accelerates or moves faster than light in a dielectric medium, it radiates. However, sources of finite size can be designed whose peculiar structure ensures that they do not radiate under these conditions. The criterion for absence of radiation of a rigid source in free space is generalized to a dielectric medium, and applied to either oscillating or Čerenkov sources.

I. INTRODUCTION

"Let there be electricity and magnetism, and there is light!" says Feynman's personal version of Genesis[1]; and, indeed, when an electron is accelerated, it radiates. Though as human beings we appreciate this property, as physicists, we do not: Separating sense from nonsense in the equations of a radiating electron is an old dream and, as Einstein once said,the electron is a stranger in electrodynamics.[2]

Even the classical electron at rest is odd: Since like charges repel, the Coulomb field tends to make it explode (unless its "mechanical mass" is negative), and one must imagine rubber strips such as the "Poincaré stresses" to hold the charge together. Anyway, the electrostatic energy of a point charge is infinite.

If the classical point electron can accelerate, the situation grows worse: While radiating, the particle undergoes a radiation reaction that has two parts (in its rest frame). The first one, which is infinite, can be viewed as a contribution to the mass, since it goes as $d^2\mathbf{r}/dt^2$. But the second one cannot be so "renormalized" since it is proportional to $d^3\mathbf{r}/dt^3$, and can cause the charge to accelerate itself[3,4] (the so-called runaway solution).

Although quantum electrodynamics is renormalizable and powerful tools[5,6] have been devised to hide the infinities under the carpet, it nevertheless cannot yield a finite energy for the point charge, nor a satisfactory theory of an extended charge.

The old problem of building a clean model for the classical electron has motivated a search for charge distributions that do not radiate.[7] Such charges could undergo force-free accelerated motion.[8]

At present, the interest of such sources seems academic:

