Chern Number and Topological Insulators
- an Introduction

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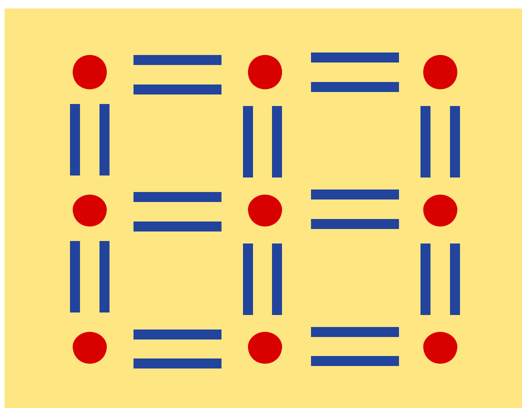
Insulators

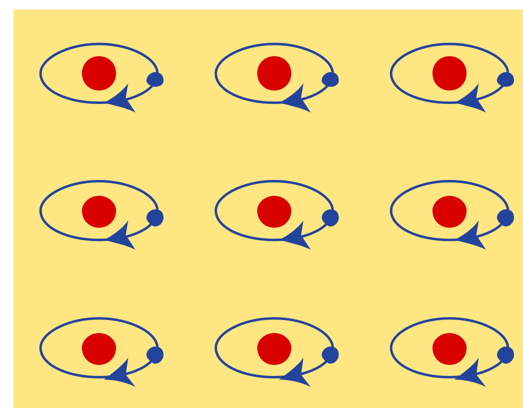
Covalent Insulator

e.g. intrinsic semiconductor

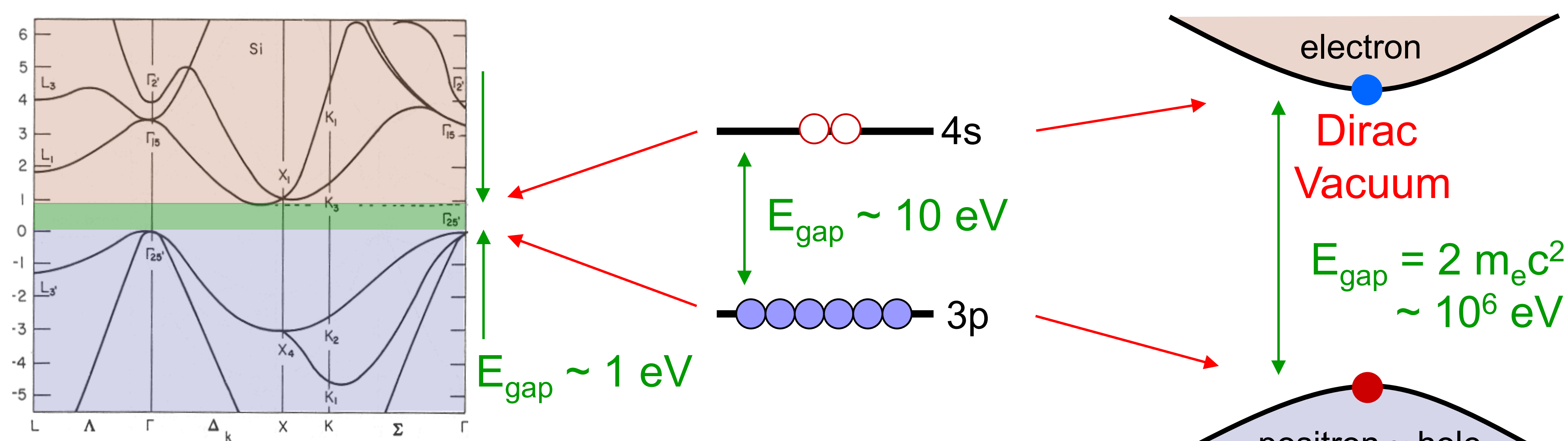


Atomic Insulator

e.g. solid Ar



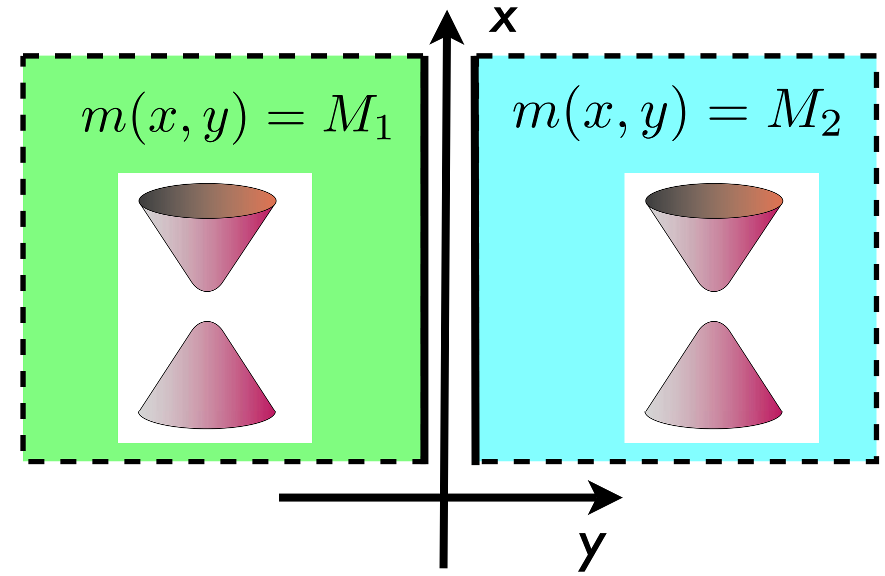
The vacuum



Silicon

Edge states in Dirac equation

- 2D 2-spinor Dirac equation $H_D\Psi = [\sigma_x p_x + \sigma_y p_y + m\sigma_z]\,\Psi(x,y) = \varepsilon\Psi(x,y)$
- Create edge by modifying parameters
- Confine electrons to edge via m(x,y)
- Keep translation invariance along x



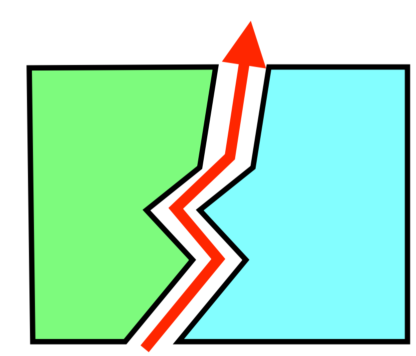
$$p_x \to q; \quad p_y \to -i\partial_y \qquad \Psi_q(x,y) = e^{iqx}\Psi(y)$$

$$-i\partial_y\Psi(y) = \underbrace{[iq\sigma_z - i\sigma_x m(y) + \varepsilon\sigma_y]}_{\mathcal{A}(y)}\,\Psi(y)$$

- General solution: $\Psi(y) = \mathbb{Y}e^{i\int_0^y \mathcal{A}y'dy'}\Psi(0)$
- Ansatz: $\boxed{\varepsilon = q}$ $\mathcal{A}(y) = iq\underbrace{(\sigma_z - i\sigma_y)}_{\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}} - im(y)\sigma_x$ $\Psi(y) = \underbrace{e^{\int_0^y m(y')dy'}}\begin{pmatrix}1\\1\end{pmatrix}$

Normalizing prefactor, if
$M_1 > 0; \quad M_2 < 0$

- Unidirectional propagation (only +x)
- Robust: no scattering to bulk (gap)
  no backscattering (unidirectional)
- Has to go across sample (unitarity)



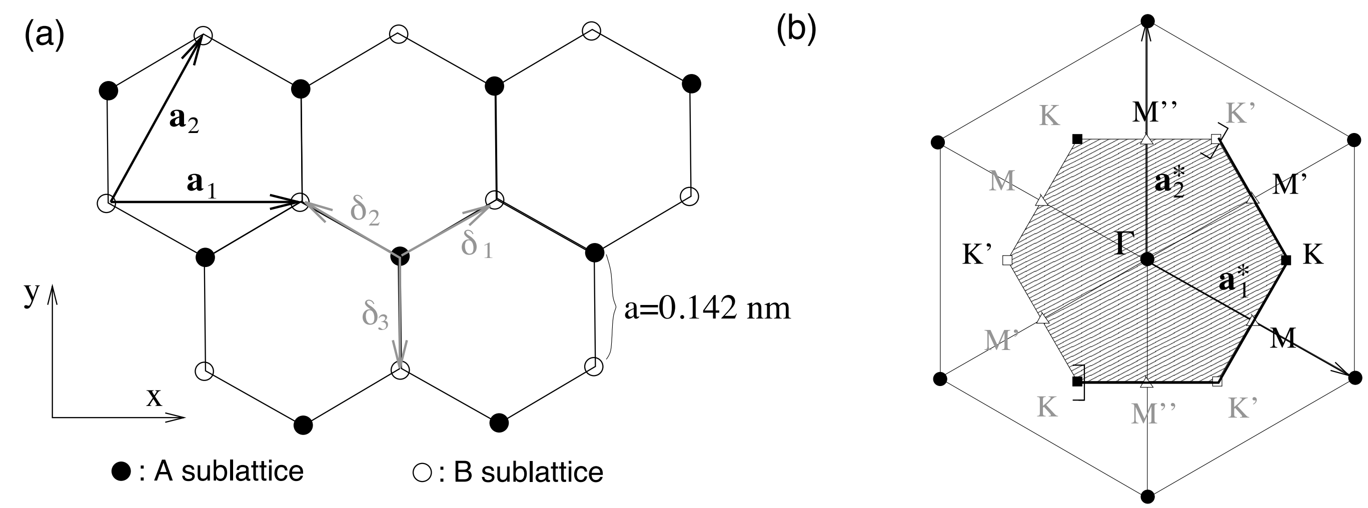
- If $M_1 < 0; \quad M_2 > 0$: unidirectional (-x)

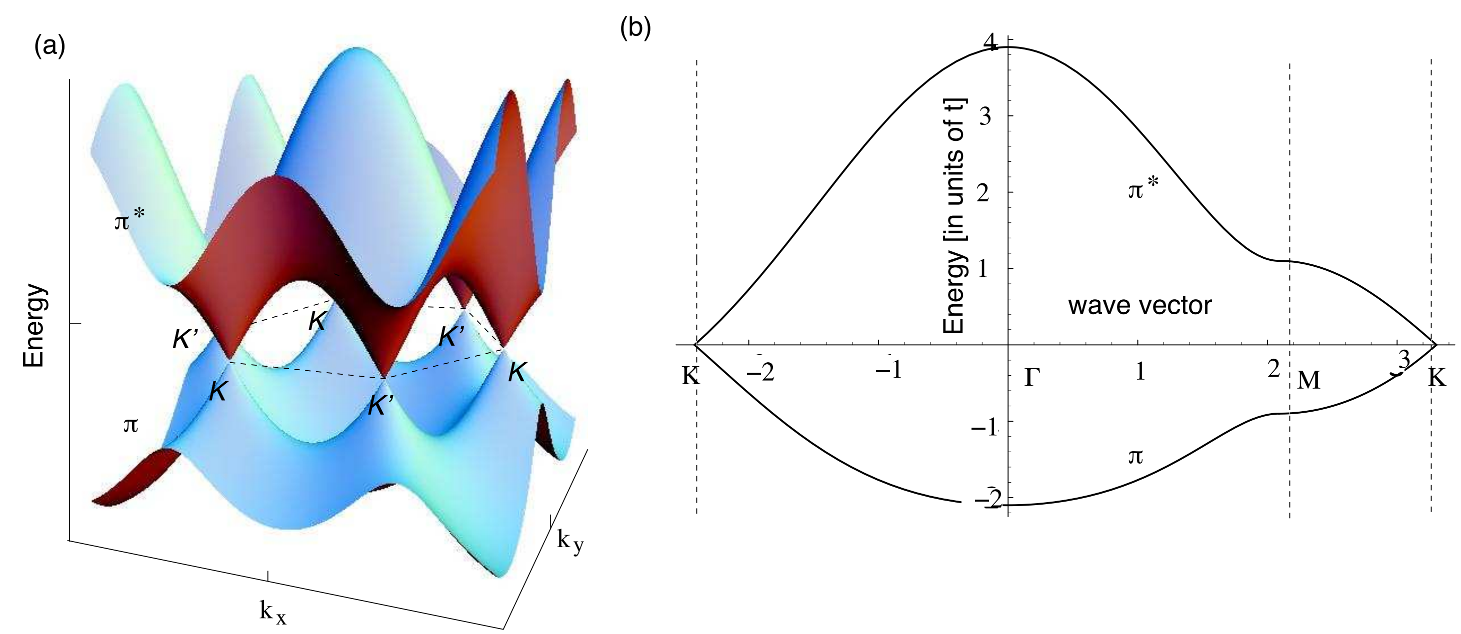
Lattice systems: several Dirac equations

- 1 electron/atom, 2 atoms/cell:
- pseudospin (sublattice)
- Gap closes at corners of Brillouin Zone (6/3=2 Dirac points)
- Low energy: 2 copies of massless Dirac equation -> 2 valleys, τ_z independent if no short-range scatterers

$$H = \tau_z p_x \sigma_x + p_y \sigma_y$$

- Induce mass (gap) via
  - substitution, Boron Nitride
  - electric field in bilayer graphene
- Counterpropagating edge states from opposite valleys

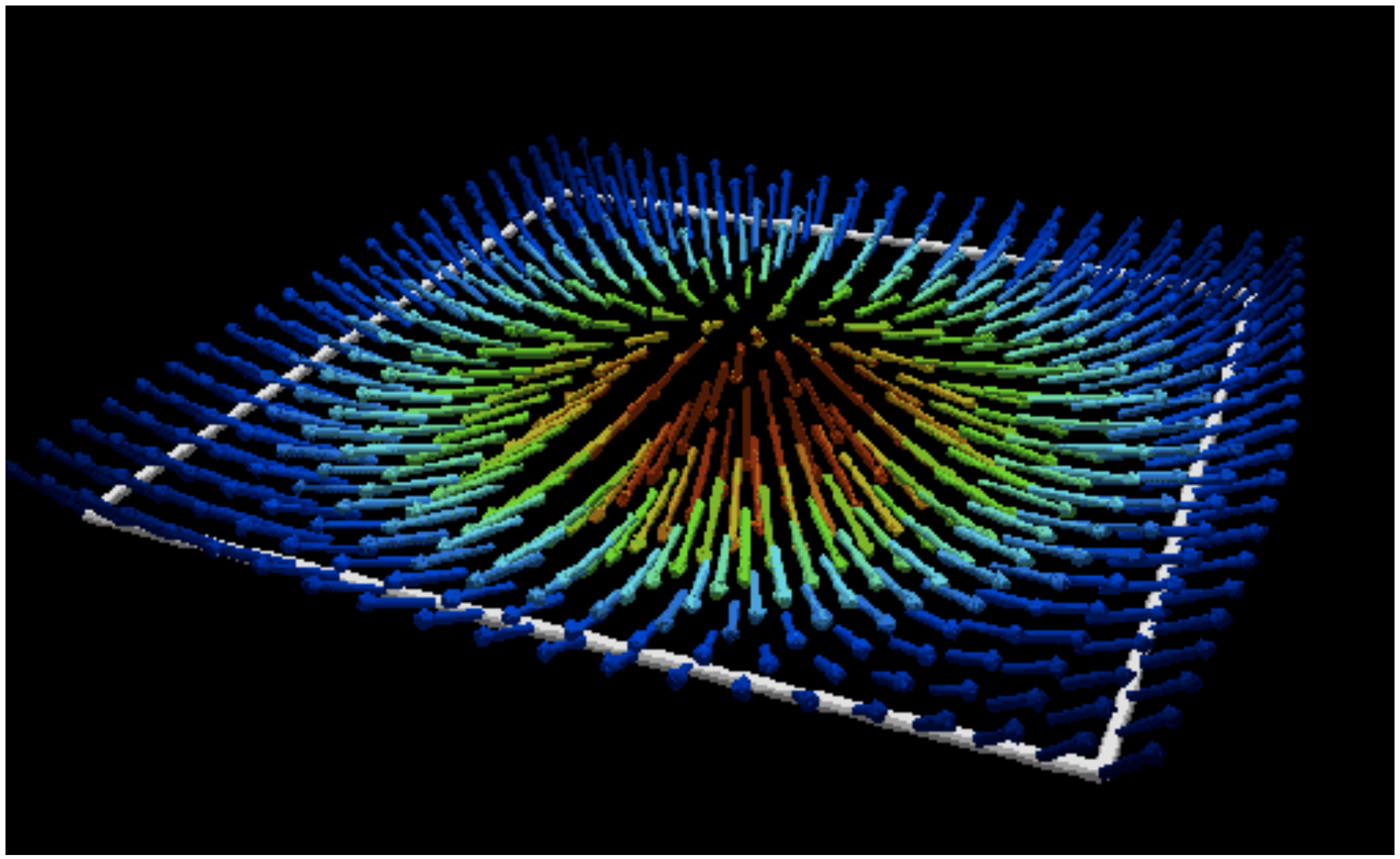




$$H = \tau_z p_x \sigma_x + p_y \sigma_y + m \sigma_z$$

Summing over valleys: Chern number

Momentum eigenstates have definite spin, 2D Brillouin zone
- Chern number: # of skyrmions in the Brillouin zone



n(**k**) maps from Brillouin zone (torus) to unit sphere:
- Chern number: # of times the sphere is covered by mapping

$$c_n = \frac{1}{4\pi}\int d^2\vec{k}\big(\partial_{k_x}\vec{n}\times\partial_{k_y}\vec{n}\big)\cdot\vec{n}$$

Consider **k** an adiabatically changed parameter:
- Chern #: Integral of Berry phase around the Brillouin zone
  = integral of Berry flux in the Brillouin zone

$$A_\mu(k) = -i\langle n(k)|\frac{\partial}{\partial k_\mu}|n(k)\rangle$$

$$F_{xy}(k) = \frac{\partial A_y}{\partial k_x} - \frac{\partial A_x}{\partial k_y}$$

$$C_1 = \frac{1}{2\pi}\int dk_x dk_y F_{xy}(k)$$

efficient discretization: need only 10 or so k-points
[Fukui, Hatsugai, Suzuki, JPSJ (2005)]

Chern #: needs Time Reversal Symmetry Breaking

- Chern number always 0 if we have a global basis
- Time reversal symmetry:
  - Complex conjugation in position and preferred internal basis: $K\Psi(k) = \Psi^*(-k)$
  - k-independent internal unitary rotation:

$$\exists T \in U(N) : \forall k \in BZ : TKH(k)KT^\dagger = H(k)$$

  - Antiunitary symmetry, $\Theta = TK$

- With time reversal invariance, we always have a global basis: Chern # = 0
- Break time reversal invariance via magnetic field
- Graphene, Haldane [PRL, 1988]: staggered magnetic field, simple calculation
  - 0 average field ⇒ no need for magnetic Brillouin Zone
  - Peierls substitution only for NNN hoppings
  - Competition between conventional and topological gap