Introduction to Majorana physics

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NanoLecture

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Outline

- Majorana found in Delft
- General properties of Majorana fermions
- Topological phases
- Kitaev model : p-wave superconductor chain
- Majoranas in superconductors, a 2D example
- Quantum computation, braiding Majoranas

Majorana found in Delft

„ Ettore Majorana, a prominent Italian theoretical physicist who mysteriously disappeared 70 years ago, has been found alive and in good health. To everyone's surprise, all this time he was living in the Netherlands, in a small village of Pijnacker in South Holland. "We were totally amazed by our discovery, but there is no doubt." said Dirk Naaktdijkloper, the leader of the private investigators squad from Delft. "We even performed a sanity DNA check, and it turned out positive". Mr. Majorana explained the reason for his disappearance. "I was in love, she did not care for the equations, so I left them behind". Professor Majorana lead a quiet but good life. In the early 80′s he managed to secure a fortune by being one of the first investors into a small computer company called Microsoft. "

http://sergeyfrolov.wordpress.com/2012/04/01/majorana-found-in-delft/

General properties of Majorana fermions

- Dirac equation (1928)

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = 1$$
$$\gamma^j\gamma^k = -\gamma^k\gamma^j \text{ for } i \neq j$$

  - complex ψ field ↔ charged particle
  - particle: ψ, antiparticle: ψ^*
- Majorana equation (1937)
  - Dirac-like equation (necessary for S=1/2 particles)
  - pure imaginary γ matrices (with Clifford algebra) to satisfy $\psi=\psi^*$

$$(i\tilde{\gamma}^\mu \partial_\mu - m)\tilde{\psi} = 0)$$

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\gamma}^1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\tilde{\gamma}^2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\tilde{\gamma}^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

1) Majorana fermions are charge neutral

2) They are their own antiparticles

F. Wilczek, Majorana returns

General properties of Majorana fermions

- Ordinary (Dirac) fermion

$$\{c_i^\dagger, c_j\} = \delta_{ij} \qquad (c_j^\dagger)^2 = c_j^2 = 0$$

- Decomposition into Majorana fermions

$$c_j = (\gamma_{j1} + i\gamma_{j2})/2$$

$$\{\gamma_{i\alpha}^\dagger, \gamma_{j\beta}\} = 2\delta_{ij}\delta_{\alpha\beta} \qquad \gamma_{i\alpha}^\dagger = \gamma_{i\alpha} \qquad (\gamma_{i\alpha})^2 = 1$$

- Inverse relations

$$\gamma_{j1} = \frac{(c_j^\dagger + c_j)}{2} = \gamma_{j1}^\dagger$$

$$\gamma_{j2} = \frac{i(c_j - c_j^\dagger)}{2} = \gamma_{j2}^\dagger$$

F. Wilczek, Majorana returns

Topological phases

	Bulk	Surface/edge
Topological insulator	insulating	metallic
Topological superconductor	superconducting	metallic

„When a system has anyonic quasiparticle excitations above its ground state, it is in a topological phase of matter.”

„A system is in a topological phase if, at low temperatures and energies and long wavelengths, all observable properties – e.g., correlation functions – are invariant under smooth deformations – diffeomorphisms – of the space-time manifold in which the system lives.”

C. Nayak, 10.1103/RevModPhys.80.1083

Majoranas in 1D – Kitaev model

- 1D spinless p-wave superconductor

$$H = -\mu \sum_x c_x^\dagger c_x - \frac{1}{2} \sum_x (t c_x^\dagger c_{x+1} + \Delta e^{i\phi} c_x c_{x+1} + h.c.)$$

- Periodic boundary conditions $\longrightarrow$ bulk properties

$$H = \frac{1}{2} \sum_{k \in BZ} C_k^\dagger \mathcal{H}_k C_k, \quad \mathcal{H}_k = \begin{pmatrix} \epsilon_k & \tilde{\Delta}_k^* \\ \tilde{\Delta}_k & -\epsilon_k \end{pmatrix} \qquad C_k^\dagger = [c_k^\dagger, c_{-k}]$$

$$\epsilon_k = -t \cos k - \mu$$

$$\tilde{\Delta}_k = -i \Delta e^{i\phi} \sin k$$

- Diagonalization

$$H = \sum_{k \in BZ} E_{\text{bulk}}(k) a_k^\dagger a_k \qquad E_{\text{bulk}}(k) = \sqrt{\epsilon_k^2 + |\tilde{\Delta}_k|^2}.$$

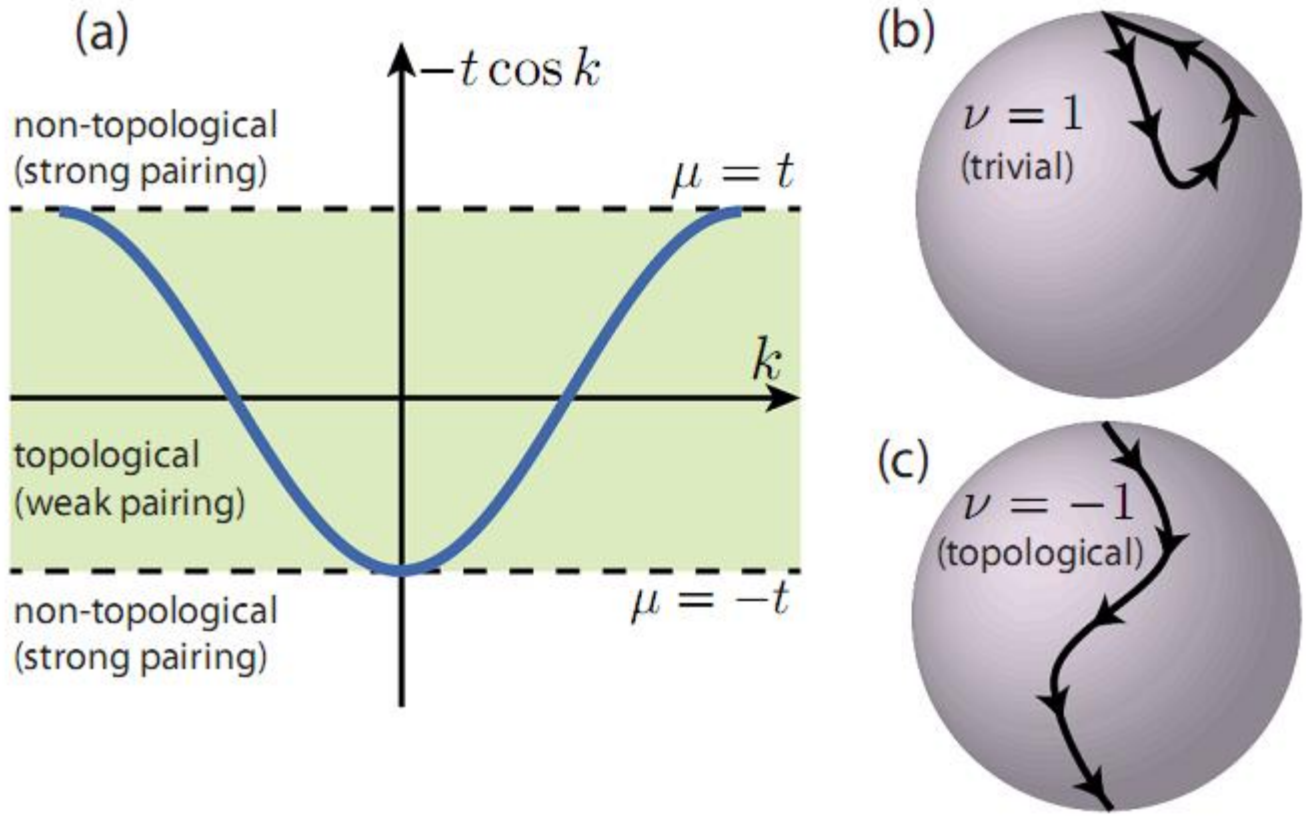
$$a_k = u_k c_k + v_k c_{-k}^\dagger$$

$$u_k = \frac{\tilde{\Delta}}{|\tilde{\Delta}|} \frac{\sqrt{E_{\text{bulk}} + \epsilon}}{\sqrt{2E_{\text{bulk}}}}, \quad v_k = \left(\frac{E_{\text{bulk}} - \epsilon}{\tilde{\Delta}} \right) u_k$$

- Analysis
  - gapless when $\mu = t$ or $-t$

$$E_{\text{bulk}}(k) = \sqrt{\epsilon_k^2 + |\tilde{\Delta}_k|^2}.$$

$$\epsilon_k = -t\cos k - \mu$$



$\mathcal{H}_k = \mathbf{h}(k)\cdot\boldsymbol{\sigma}$ $\quad$ $\hat{\mathrm{h}}(0) = s_0\hat{\mathbf{z}},\ \ \hat{\mathrm{h}}(\pi) = s_\pi\hat{\mathbf{z}}$ $\quad$ $\nu = s_0 s_\pi$

Majoranas in 1D – Kitaev model

- Majoranas at the open ends

$$c_x = \frac{e^{-i\phi/2}}{2}(\gamma_{B,x} + i\gamma_{A,x})$$

$$\gamma_{\alpha,x} = \gamma^{\dagger}_{\alpha,x}, \qquad \{\gamma_{\alpha,x}, \gamma_{\alpha',x'}\} = 2\delta_{\alpha\alpha'}\delta_{xx'}$$

⇓

$$H = -\frac{\mu}{2}\sum_{x=1}^{N}(1 + i\gamma_{B,x}\gamma_{A,x}) - \frac{i}{4}\sum_{x=1}^{N-1}[(\Delta + t)\gamma_{B,x}\gamma_{A,x+1} + (\Delta - t)\gamma_{A,x}\gamma_{B,x+1}] \quad (16)$$

Majoranas in 1D – Kitaev model

- Majoranas at the open ends

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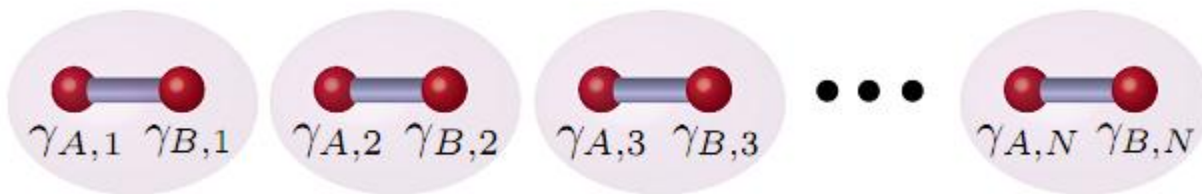
$$f = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

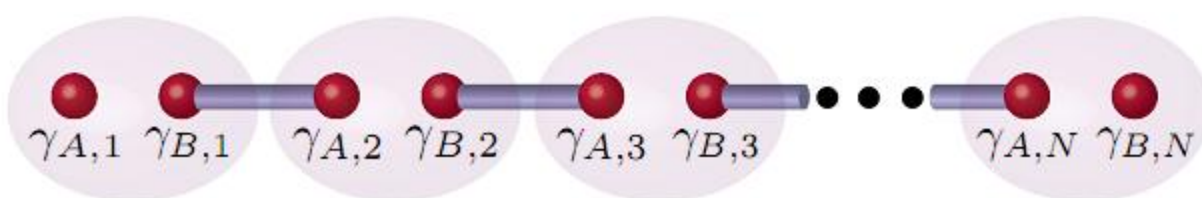
(a)

$\mu \neq 0,\ t = \Delta = 0$



(b)

$\mu = 0,\ t = \Delta \neq 0$



Majoranas in superconductors

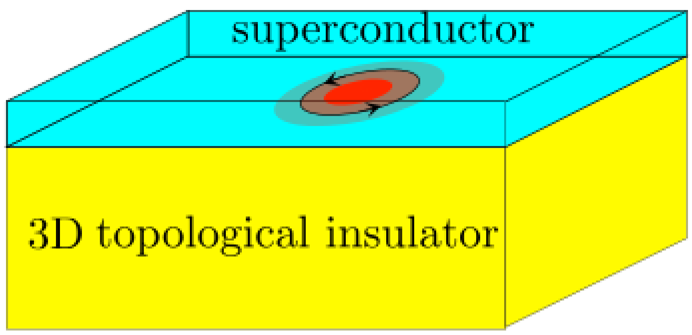
- Electron-hole symmetry

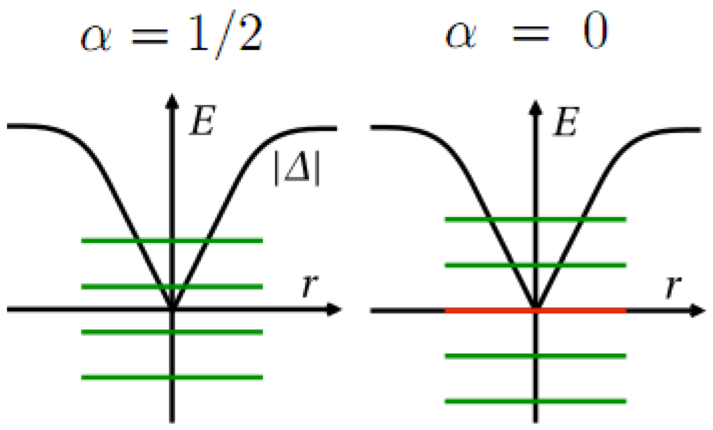
$$\gamma(E) = \gamma^\dagger(-E)$$

  - at the Fermi level: $\gamma(0) \equiv \gamma = \gamma^\dagger$
- General recipe:
  - take a superconductor
  - remove degeneracies
  - close and reopen the excitation gap
- Example: vortices on SC-TI interface
  - SC opens a gap in the TI
  - the gap can be closed by applying a magnetic field

C. W. J. Beenakker, arXiv: 1112.1950

- Example: vortices on SC-TI interface
  - SC opens a gap in the TI
  - the gap can be closed by applying a magnetic field
  - subgap states: $E_n = (n + \alpha)\delta$
  - TI: massless Dirac fermions on the surface → $\alpha = 0$





C. W. J. Beenakker, arXiv: 1112.1950

Exchange of indistinguishable particles

- 3D

$$\psi \mapsto \pm\psi$$

- 2D

a) $$\psi \mapsto e^{i\theta}\psi$$

$$\theta = 0 \rightarrow \text{boson}$$
$$\theta = \pi \rightarrow \text{fermion}$$
$$\theta \neq 0,\ \theta \neq \pi \rightarrow \text{anyon with fractional statistics}$$

} Abelian statistics

b) $$\psi \mapsto U\psi$$

Anyon with non-Abelian statistics

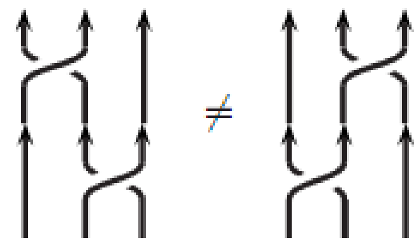
- 1D
  – quantum statistics is not well defined

C. Nayak, 10.1103/RevModPhys.80.1083
J. Alicea, arXiv:1202.1293

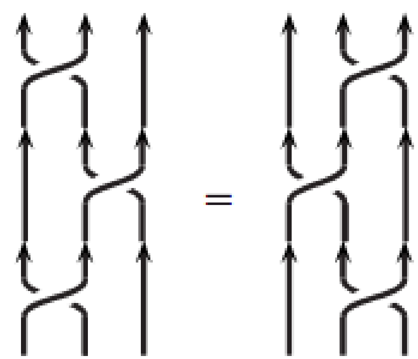
Quantum computation – braiding of Majoranas

- Braiding = adiabatic interchange of identical particles $\psi \mapsto U\psi$
- Braid group $\mathcal{B}_N$
  - defining relation

    $\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| \geqslant 2 \qquad \sigma_i^2 \neq 1$
  - similiar to permutation group S_N, but the braid group is infinite even for 2 particles, while $|S_N| = N!$
- „[…] a process in which two particles are adiabatically interchanged twice is equivalent to a process in which one of the particles is adiabatically taken around the other."



$\sigma_2\sigma_1 \neq \sigma_1\sigma_2$



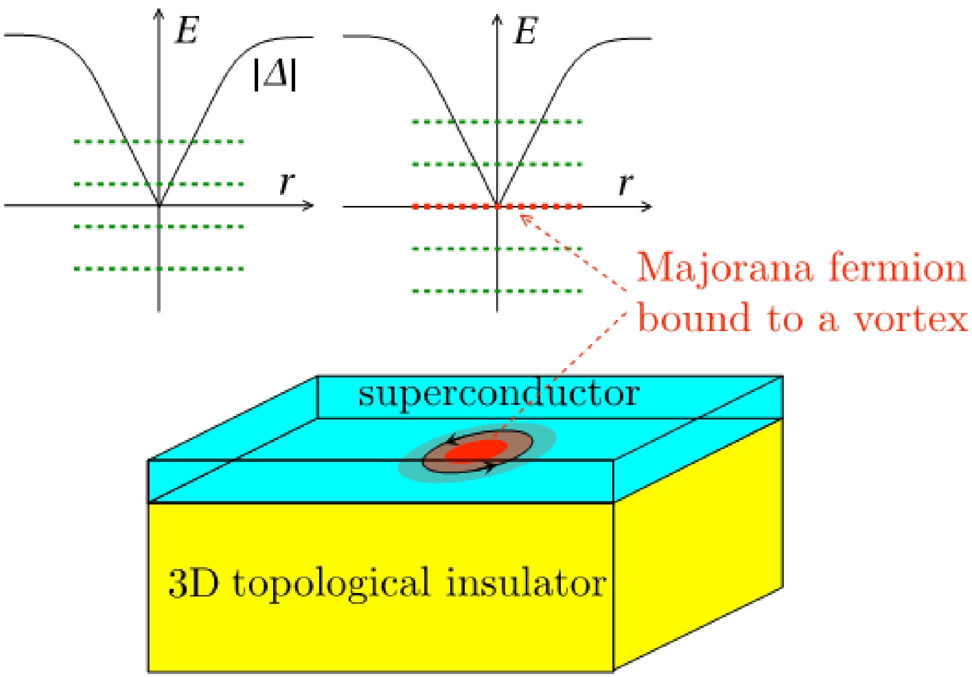
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

braid relation

C. Nayak, 10.1103/RevModPhys.80.1083

- Braiding in 2D
  - 2N vortices on the SC-TI interface with Majorana modes $\gamma_{1,\ldots,2N}$
  - N fermion operators with E=0 $f_j = (\gamma_{2j-1} + i\gamma_{2j})/2$
  - 2^N degenerate ground states: $|n_1, n_2, \ldots, n_N\rangle$

$$n_j = f_j^\dagger f_j$$



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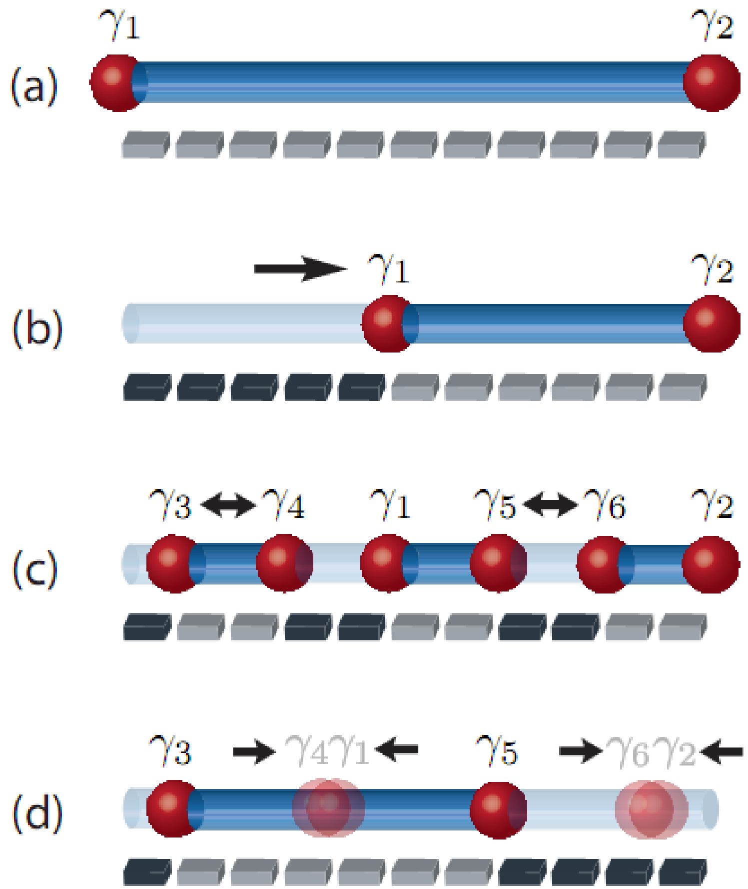
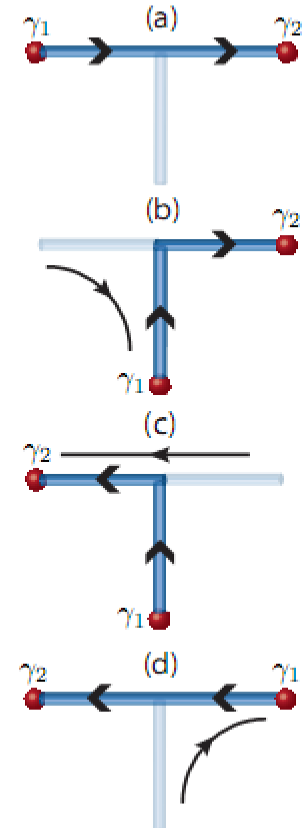
- Braiding in 1D
  - 1D wire, control of topological phase with a keyboard
  - network of 1D wires, T junction





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Summary

- Majorana fermions
  - as elementary particles, particle physics
  - as excitations in the solid state
- Platform for quantum computation
  - Non-Abelian exchange statistics
  - Resistance to local perturbations